Journal of Sound and Vibration (1998) **209**(5), 771–796

FREE VIBRATION AND BUCKLING OF A PARTIALLY SUBMERGED CLAMPED CYLINDRICAL TANK UNDER COMPRESSION

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(*Received* 10 *December* 1996, *and in final form* 12 *August* 1997)

Theoretical analysis and experimental study have been carried out on the free vibration and buckling under axial compression of a clamped–clamped cylindrical shell which partially contains liquid and is partially submerged in a liquid. In the analysis, the thin elastic shell is assumed to be submerged in a rigid cylindrical container with finite diameter. Considering the effect of the static liquid pressures inside and outside the shell, coupled bulging-type natural frequencies and critical axial load parameters were calculated for some system parameters, i.e., the thickness ratio, the aspect ratio, the liquid heights, and compressive load parameter. The effects of liquid height both outside and inside the shell, and static compressive load, on the bulging-type natural frequency, were clarified. The results are summarized in the form of engineering design data from which one can easily predict the natural frequency and the critical load of a given tank partially submerged in a liquid and containing a liquid. To confirm the accuracy of the theoretical analysis, an experimental study was conducted on a test cylinder made of polyester film. On the natural frequency, excellent agreement between theoretical and experimental results was demonstrated. Some results were compared with those of a clamped–free shell to see the influence of the boundary condition.

1. INTRODUCTION

As one of the fundamental structural components in atomic plant systems and chemical plants, and so on, liquid containing or liquid faced cylindrical structures have been widely used. Submerged cylindrical structures have also been used in ocean development plants to create spaces for living, working, and storage. Hence, it is very important to clarify the buckling strength under various loads, i.e., compressive, torsional, dynamic and static loads in order to conduct safety design for earthquakes and wave forces, and to clarify the vibration characteristics.

The buckling problem for a cylindrical shell has been studied by many researchers, and the problems for a simple load, i.e. axial compression, external pressure, and torsion has already been clarified. For problems under combined loads some studies have been conducted. In 1982, Doki *et al*. studied the buckling problem of a liquid filled cylindrical shell under external pressure [1] and compression [2]. They considered the effect of prebuckling deformation, and clarified the influence of an inner liquid on the buckling. In 1983, Kodama and Yamaki studied the problem under inner and outer pressures, and under axial load [3]. In 1989, Chiba *et al*. treated the buckling problem on a cylindrical

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shell partially submerged in a liquid theoretically and experimentally [4]. They treated both a clamped–free and a clamped–clamped shell. In 1996, Chiba and Ubukata studied the influence of an inner liquid on the buckling of a cylindrical shell partially submerged in a liquid [5].

For the free vibration problem, theoretical and experimental studies have been conducted on a partially liquid containing shell considering the effect of static liquid pressure in the analysis by Yamaki *et al*. for a clamped–clamped case [6] and by Chiba *et al*. for a clamped–free case [7–9]. For a shell submerged in a liquid, Chiba conducted a theoretical analysis and experiment for a clamped–free shell [10] and for a clamped–free tank partially containing liquid [11].

The aim of the present study is to clarify the free vibration characteristics and the buckling strength under compressive axial load of a partially liquid containing and partially submerged cylindrical shell with clamped–clamped boundary condition. This problem has never been treated before, to the best of the authors' knowledge. For the natural frequencies and the buckling loads, the computed results were normalized by those of the liquidless shell. To see the influence of the boundary condition, the results for clamped–free shells are also presented. To confirm the validity of the analysis, experimental studies were also conducted for the natural frequency of a test cylinder made of polyester film.

2. FORMULATION OF THE PROBLEM

2.1. LIQUID CONTAINED SUBMERGED TANK

Let us consider the linear free vibration and the stability of a thin perfect cylindrical shell with radius R, length L and thickness h, which is submerged in a rigid cylindrical container with radius R_o to a height H_o, and filled with liquid to a height H_i (Figure 1). The shell is assumed to be isotropic and has a clamped–clamped boundary condition, while the inner and outer liquids are inviscid, incompressible, and have the same density. The compressive load $P = 2\pi R h \sigma_c$ is applied in the axial direction of the shell. Confining the problem to relatively low-frequency ranges dominated by flexural motion of the wall, we will apply the Donnell shell theory. Defining the co-ordinate as shown in Figure 1, the

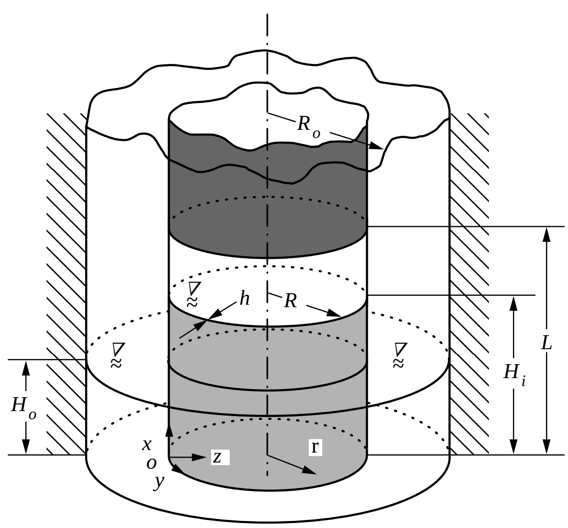

Figure 1. Liquid contained submerged clamped–clamped cylindrical tank under compression.

deformation components of the middle plane of the shell are $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ in the x, y, z directions, and the stress resultants $\tilde{N}_x$, $\tilde{N}_y$, $\tilde{N}_{xy}$, respectively, the governing equations of the shell are

$$\tilde{N}_x,_x + \tilde{N}_{xy},_y = 0, \qquad \tilde{N}_{xy},_x + \tilde{N}_y,_y = 0, \tag{1}$$

$$D\nabla^4\tilde{W} - \frac{1}{R}\tilde{N}_y - (\tilde{N}_x\,\tilde{W},_{xx} + 2\tilde{N}_{xy}\,\tilde{W},_{xy} + \tilde{N}_y\,\tilde{W},_{yy}) + \rho_s h\tilde{W},_{tt} + P_i + P_o = 0, \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{3}$$

In these equations, E, D and ν denote Young's modulus, flexural rigidity, and Poisson's ratio of the shell. P_i and P_o are liquid pressures of the inner and the outer liquids.

The shell is subjected to static liquid pressure from both the inner and the outer liquid in the z direction, and compressive load in the x direction, which produce the axisymmetric static deformation of the wall. Then, the shell undergoes small amplitude vibration around the axisymmetric state. The inner and the outer liquid pressures, P_i and P_o, are sub-divided into the static and dynamic components

$$P_j = P_{sj} + P_{dj} \quad (j = i, o). \tag{4}$$

Assuming the liquid to be irrotational, we can introduce the velocity potential $\Phi_j\,(x, y, r, t)$, $j = i, o$, from which we can obtain the dynamic pressure component as

$$P_{si} = \rho_f\,g(H_i - x) = \rho_f\,gH_i\,(1 - (x/H_i))\epsilon_i,$$

$$P_{so} = -\rho_f\,g(H_o - x) = -\rho_f\,gH_o\,(1 - (x/H_o))\epsilon_o,$$

$$P_{di} = -\rho_f\,[\Phi_i,_t]_{r=R}\,\epsilon_i, \qquad P_{do} = \rho_f\,[\Phi_o,_t]_{r=R}\,\epsilon_o, \qquad \epsilon_j = \begin{cases} 0 : H_j < x \leqslant L \\ 1 : 0 < x \leqslant H_j \end{cases} \quad (j = i, o), \tag{5}$$

where the subscript $j = i, o$ corresponds to the inner and the outer liquid, respectively.

The relations between the deformations and the stress resultants are

$$Eh(\tilde{U},_x + 1/2\,\tilde{W},_x^2) = \tilde{N}_x - \nu\tilde{N}_y,$$

$$Eh\left(\tilde{V},_y - \frac{\tilde{W}}{R} + \frac{1}{2}\tilde{W},_y^2\right) = \tilde{N}_y - \nu\tilde{N}_x,$$

$$Eh(\tilde{U},_y + \tilde{V},_x + \tilde{W},_x\,\tilde{W},_y) = 2(1+\nu)\tilde{N}_{xy}. \tag{6}$$

Introducing the stress function $\tilde{F}$,

$$\tilde{N}_x = \tilde{F},_{yy}, \qquad \tilde{N}_y = \tilde{F},_{xx}, \qquad \tilde{N}_{xy} = -\tilde{F},_{xy}, \tag{7}$$

equation (1) is satisfied. Eliminating $\tilde{U}$ and $\tilde{V}$ from equation (6), one obtains

$$\nabla^4\tilde{F} + Eh\left(\frac{1}{R}\tilde{W},_{xx} + \tilde{W},_{xx}\,\tilde{W},_{yy} - \tilde{W},_{xy}^2\right) = 0. \tag{8}$$

By using equations (7), (2) and (6),

$$D\nabla^4\tilde{W} - \frac{1}{R}\tilde{F},_{xx} - \tilde{F},_{yy}\,\tilde{W},_{xx} + 2\tilde{F},_{xy}\,\tilde{W},_{xy} - \tilde{F},_{xx}\,\tilde{W},_{yy} + \rho_s h\tilde{W},_{tt} + P_i + P_o = 0, \tag{9}$$

$$Eh(\tilde{U}_{,x} + 1/2\ \tilde{W}_{,x}^2) = \tilde{F}_{,yy} - \nu\tilde{F}_{,xx}. \tag{10}$$

The shell is clamped at the boundaries

$$x = 0, L: \quad \tilde{W} = \tilde{W}_{,x} = 0,$$
$$\tilde{U}_{,y} = \tilde{V}_{,y} = 0,$$

or

$$\tilde{F}_{,xxx} + (2+\nu)\tilde{F}_{,xyy} = \tilde{F}_{,xx} - \nu\tilde{F}_{,yy} = 0, \tag{11}$$

and the following equations must be satisfied

$$\int_{-\pi R}^{\pi R} \tilde{N}_x \, \mathrm{d}y = -2\pi R\sigma_c h, \qquad \int_{-\pi R}^{\pi R} \tilde{N}_{xy} \, \mathrm{d}y = 0. \tag{12}$$

Deformation components and stress resultants can be sub-divided into static and dynamic ones as,

$$\tilde{U} = U_0 + U, \qquad \tilde{V} = V_0 + V, \qquad \tilde{W} = W_0 + W, \qquad \tilde{N}_x = N_{x0} + N_x, \qquad \tilde{N}_y = N_{y0} + N_y,$$
$$\tilde{N}_{xy} = N_{xy0} + N_{xy}, \qquad \tilde{F} = F_0 + F. \tag{13}$$

2.1.1. *Axisymmetric deformation due to static liquid pressure and compressive load*

Substituting equation (13) into equations (8)–(12), we obtain

$$F_{0,xy} = 0, \qquad V_0 = 0, \tag{14}$$

$$F_{0,xxxx} + \frac{Eh}{R} W_{0,xx} = 0, \tag{15}$$

$$DW_{0,xxxx} - \frac{1}{R} F_{0,xx} - F_{0,yy} W_{0,xx} + P_{si} + P_{so} = 0, \tag{16}$$

$$Eh\left(U_{0,x} + 1/2\ W_{0,x}^2\right) = F_{0,yy} - \nu F_{0,xx}, \tag{17}$$

$$-Eh\frac{W_0}{R} = F_{0,xx} - \nu F_{0,yy}. \tag{18}$$

$$x = 0, L: \quad W_0 = W_{0,x} = 0, \tag{19}$$

$$\int_{-\pi R}^{\pi R} N_{x0} \, \mathrm{d}y = -2\pi R\sigma_c h, \qquad \int_{-\pi R}^{\pi R} N_{xy0} \, \mathrm{d}y = 0. \tag{20}$$

2.1.2. *Small amplitude asymmetric vibration around the axisymmetric state*

Furthermore, considering the linear terms of U, V and W, we obtain

$$\nabla^4 F + Eh\left(\frac{1}{R} W_{,xx} + W_{0,xx} W_{,yy}\right) = 0, \tag{21}$$

$$D\nabla^4 W - \frac{1}{R} F_{,xx} - F_{,yy} W_{0,xx} - F_{0,yy} W_{,xx} - F_{0,xx} W_{,yy} + \rho_s h W_{,tt} + P_{di} + P_{do} = 0, \tag{22}$$

$$Eh(U_{,x} + W_{0,x} W_{,x}) = F_{,yy} - \nu F_{,xx},$$

$$Eh\left(V_{,y} - \frac{W}{R}\right) = F_{,xx} - \nu F_{,yy},$$

$$Eh(U_{,y} + V_{,x} + W_{0,x} W_{,y}) = -2(1+\nu)F_{,xy}. \tag{23}$$

$$x = 0, L: \quad W = W_{,x} = 0,$$

$$F_{,xxx} + (2+\nu)F_{,xyy} = F_{,xx} - \nu F_{,yy} = 0. \tag{24}$$

2.1.3. *Motions of the liquids*

Velocity potential $\Phi_j\,(x, y, r, t)$, $j = i, o$, must satisfy the following Laplace equation,

$$\Phi_{j,rr} + \frac{1}{r}\Phi_{j,r} + \frac{1}{r^2}\Phi_{j,\theta\theta} + \Phi_{j,xx} = 0, \qquad j = i, o. \tag{25}$$

The boundary conditions are as follows:
The velocity at the bottom is zero:

$$x = 0: \quad \Phi_{j,x} = 0. \tag{26}$$

Free surface conditions:

$$x = H_j: \quad \Phi_{j,tt} + g\Phi_{j,x} = 0. \tag{27}$$

The velocity matching conditions at the liquid-shell interface:

$$r = R, \qquad 0 < x < H_j: \quad \Phi_{j,r} = -W_{,t}. \tag{28}$$

The velocity at the outer rigid wall is zero:

$$r = R_o: \quad \Phi_{o,r} = 0. \tag{29}$$

2.2. NON-DIMENSIONALIZATION

Here, we introduce the following non-dimensional parameters:

$$\xi = \frac{\pi x}{L}, \qquad \eta = \frac{y}{R}N, \qquad \beta = \frac{LN}{\pi R}, \qquad \tilde{\beta} = \frac{\beta}{N}, \qquad \alpha = \frac{L^2}{\pi^2 Rh}, \qquad \tau = \Omega_o t, \qquad f = \frac{F}{Eh^3},$$

$$\tilde{p}_o = \frac{L^5 g\rho_f}{\pi^4 h^4 E}, \qquad k_c = \frac{\sigma_c hL^2}{\pi^2 D}, \qquad (u_o, u) = \frac{L}{\pi h^2}(U_o, U), \qquad (v_o, v) = \frac{L}{\pi h^2}(V_o, V),$$

$$(w_o, w) = (W_o, W)/h, \qquad (l_i, l_o) = (H_i, H_o)/L, \qquad \gamma = \frac{12L^4\rho_f}{\pi^4 Rh^3\rho_s},$$

$$\Omega_o = \frac{1}{R}\sqrt{\frac{E}{\rho_s(1-\nu^2)}}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \qquad \omega = \frac{\Omega}{\Omega_o}, \qquad c = \frac{1}{12(1-\nu^2)},$$

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial\xi^2} + \beta^2\frac{\partial^2}{\partial\eta^2}, \qquad \phi = \frac{\Phi}{Rh\Omega_o}, \qquad \bar{g} = \frac{g\pi}{L\Omega_o^2}, \qquad \bar{\bar{g}} = \frac{gh\rho_s}{E}, \qquad \rho = \frac{r}{R},$$

$$\bar{\rho} = \frac{\rho_s}{\rho_f}, \qquad \rho_o = \frac{R_o}{R}, \qquad \bar{R} = \frac{R}{h}, \qquad \epsilon_j = \begin{cases} 0: & \pi l_j < \xi \leqslant \pi \\ 1: & 0 < \xi \leqslant \pi l_j \quad j = i, o. \end{cases} \tag{30}$$

In these equations, t is time, g is the gravitational acceleration, Ω is the unknown vibration frequency, and N is the circumferential wave number of the vibration of the shell. l_i and

l_o are non-dimensional parameters related to the liquid heights H_i and H_o, respectively. Here, arranging the system parameters, we have thickness ratio R/h, aspect ratio L/R, density ratio $\bar{\rho}$, radius ratio ρ_o, material parameter $\bar{\bar{g}}$, compressive load parameter k_c and liquid heights l_i and l_o.

The governing equations and the boundary conditions of the liquid-shell system are expressed as follows.

2.2.1. *Axisymmetric deformation*

$$f_0,_{\xi\xi\xi\xi} + \alpha w_0,_{\xi\xi} = 0, \tag{31}$$

$$w_0,_{\xi\xi\xi\xi} - \frac{\alpha}{c} f_0,_{\xi\xi} - \frac{\beta^2}{c} f_0,_{\eta\eta}\, w_0,_{\xi\xi} + \frac{\tilde{p}_o}{c}\left[l_i\left(1 - \frac{\xi}{\pi l_i}\right)\epsilon_i - l_o\left(1 - \frac{\xi}{\pi l_o}\right)\epsilon_o \right] = 0, \tag{32}$$

$$u_0,_{\xi} + 1/2\, w_0^2,\xi = \beta^2 f_0,_{\eta\eta} - v f_0,_{\xi\xi}, \tag{33}$$

$$-\alpha w_0 = f_0,_{\xi\xi} - v\beta^2 f_0,_{\eta\eta}. \tag{34}$$

$$\xi = 0, \pi: \quad w_0 = w_0,_{\xi} = 0, \tag{35}$$

$$f_0,_{\eta\eta} = -\frac{c}{\beta^2} k_c, \qquad f_0,_{\xi\eta} = 0. \tag{36}$$

2.2.2. *Small amplitude asymmetric vibration*

$$\bar{\nabla}^4 f + \alpha w,_{\xi\xi} + \beta^2 w_0,_{\xi\xi}\, w,_{\eta\eta} = 0, \tag{37}$$

$$\bar{\nabla}^4 w - \frac{\alpha}{c} f,_{\xi\xi} - \frac{\beta^2}{c} f,_{\eta\eta}\, w_0,_{\xi\xi} - \frac{\beta^2}{c} f_0,_{\eta\eta}\, w,_{\xi\xi} - \frac{\beta^2}{c} f_0,_{\xi\xi}\, w,_{\eta\eta} + 12\alpha^2 w,_{\tau\tau} + p_{di} + p_{do} = 0, \tag{38}$$

$$p_{di} = -\gamma\phi_i,_{\tau}\,|_{\rho=1}\cdot\epsilon_i, \qquad p_{do} = \gamma\phi_o,_{\tau}\,|_{\rho=1}\cdot\epsilon_o, \tag{39}$$

$$u,_{\xi} + w_0,_{\xi}\, w,_{\xi} = \beta^2 f,_{\eta\eta} - v f,_{\xi\xi},$$

$$\beta v,_{\eta} - \alpha w = f,_{\xi\xi} - v\beta^2 f,_{\eta\eta},$$

$$\beta u,_{\eta} + v,_{\xi} + \beta w_0,_{\xi} w,_{\eta} = -2(1+v)\beta f,_{\xi\eta}. \tag{40}$$

$$\xi = 0, \pi: \quad w = w,_{\xi} = 0, \tag{41}$$

$$f,_{\xi\xi\xi} + (2+v)\beta^2 f,_{\xi\eta\eta} = f,_{\xi\xi} - v\beta^2 f,_{\eta\eta} = 0. \tag{42}$$

2.2.3. *Motions of the liquids*

$$\phi_j,_{\rho\rho} + \frac{1}{\rho}\phi_j,_{\rho} + \left(\frac{N}{\rho}\right)^2 \phi_j,_{\eta\eta} + \left(\frac{1}{\bar{\beta}}\right)^2 \phi_j,_{\xi\xi} = 0, \qquad j = i, o. \tag{43}$$

$$\xi = 0: \quad \phi_j,_{\xi} = 0, \tag{44}$$

$$\xi = \pi l_j: \quad \phi_j,_{\tau\tau} + \bar{g}\phi_j,_{\xi} = 0, \tag{45}$$

$$\rho = 1, 0 < \xi < \pi l_j: \quad \phi_j,_{\rho} = -w_{\tau}, \tag{46}$$

$$\rho = \rho_o: \quad \phi_o,_{\rho} = 0. \tag{47}$$

2.2.4. *Axisymmetric vibration*

For the axisymmetric free vibration, the governing equation and the boundary conditions of the shell are:

$$w,_{\xi\xi\xi\xi} + \frac{\alpha^2}{c} w - \frac{\beta^2}{c} f_{0},_{\eta\eta} w,_{\xi\xi} + 12\alpha^2 w,_{\tau\tau} + p_{di} + p_{do} = 0. \tag{48}$$

$$\xi = 0, \pi: \quad w = w,_{\xi} = 0. \tag{49}$$

As for the liquid, the governing equation is

$$\phi_{j},_{\rho\rho} + \frac{1}{\rho} \phi_{j},_{\rho} + \left(\frac{1}{\tilde{\beta}}\right)^2 \phi_{j},_{\xi\xi} = 0, \qquad j = i, o. \tag{50}$$

and the boundary conditions are given by equations (44–47).

3. METHOD OF SOLUTION

The problem is reduced to an eigenvalue problem to find a coupled natural frequency as an eigenvalue and to find a critical compressive load in the following manner.

3.1. AXISYMMETRIC DEFORMATION DUE TO STATIC LIQUID PRESSURE AND COMPRESSIVE LOAD

Considering that the compressive load is uniform in the ξ direction and stress resultants are uniform in the η direction, we get from equations (31) and (34),

$$f_{o},_{\xi} = -\alpha w_0 - \nu c k_c, \qquad f_{0},_{\eta\eta} = -\frac{c}{\beta^2} k_c, \qquad f_{0},_{\xi\eta} = 0, \tag{51}$$

and

$$L_0(w_0) \equiv w_{0},_{\xi\xi\xi\xi} + k_c\, w_{0},_{\xi\xi} + \frac{\alpha^2}{c} w_0 + \frac{\tilde{p}_o}{c}\left[l_i \left(1 - \frac{\xi}{\pi l_i}\right)\epsilon_i - l_o\left(1 - \frac{\xi}{\pi l_o}\right)\epsilon_o \right] + \nu \alpha k_c = 0. \tag{52}$$

Considering the boundary condition (35), $w_0(\xi)$ is assumed in the form as

$$w_0(\xi) = \sum_n a_n \psi_n(\xi), \qquad n = 1, 2, 3, \ldots, \tag{53}$$

where the a_n are unknown constants and $\psi_n(\xi)$ are the eigenfunctions of clamped–clamped beams which are defined as

$$\psi_n(\xi) = \mu_n(\cosh \alpha_n \xi - \cos \alpha_n \xi) - \nu_n(\sinh \alpha_n \xi - \sin \alpha_n \xi), \tag{54}$$

$$\mu_n = \frac{(\cosh \alpha_n \pi - \cos \alpha_n \pi)}{\kappa_n}, \qquad \nu_n = \frac{(\sinh \alpha_n \pi + \sin \alpha_n \pi)}{\kappa_n}, \tag{55}$$

$$\kappa_n = \sqrt{\pi} \cdot \sinh \alpha_n \pi \cdot \sin \alpha_n \pi, \tag{56}$$

where $\alpha_n \pi$ are the parameters which satisfy

$$1 - \cos \alpha_n \pi \cdot \cosh \alpha_n \pi = 0. \tag{57}$$

Substituting equation (53) into equation (52), and applying the Galerkin method,

$$\int_0^{\pi} L_0(w_0) \cdot \psi_l(\xi)\, d\xi = 0, \qquad l = 1, 2, 3, \ldots, \tag{58}$$

from which one can obtain a coupled linear equation in terms of a_n.

$$\sum_n a_n \begin{bmatrix} \dfrac{k_c}{\alpha_n^4 - \alpha_l^4}(A_{nl} - A_{ln}): & n \neq l \\ k_c\, \alpha_l \{q_l\,(\alpha_l\, q_l\, \pi^2 - 1) - v_l\, \mu_l\} - \left(\alpha_l^4 + \dfrac{\alpha^2}{c}\right): & n = l \end{bmatrix}$$

$$= \frac{\tilde{p}_o}{c\alpha_l^4\, \pi}\{\psi_l''(\pi l_i) - \psi_l''(\pi l_o) + 2\alpha_l^3\, v_l\,(\pi l_i - \pi l_o)\} + \frac{v\alpha k_c}{\alpha_l^4}\{\psi_l''(\pi) + 2\alpha_l^3\, v_l\}, \tag{59}$$

where

$$A_{nl} = 4\alpha_n^2\, \alpha_l^3\,(q_l + v_l\, \mu_n), \qquad q_n = \frac{1}{\pi}(\cot \alpha_n \pi + \coth \alpha_n \pi). \tag{60}$$

3.2. ASYMMETRIC FREE VIBRATION AROUND THE AXISYMMETRIC DEFORMATION

3.2.1. *Stress function*

Next, small amplitude asymmetric vibration around the axisymmetric deformed state, with circumferential wave number $N(>1)$ is considered. We assume shell deflection w and the corresponding stress function f in the form

$$w(\xi, \eta, \tau) = \mathrm{e}^{i\omega\tau} \cos \eta \sum_m b_m\, \psi_m(\xi), \tag{61}$$

$$f(\xi, \eta, \tau) = \alpha\, \mathrm{e}^{i\omega\tau} \cos \eta\, \bar{f}(\xi), \tag{62}$$

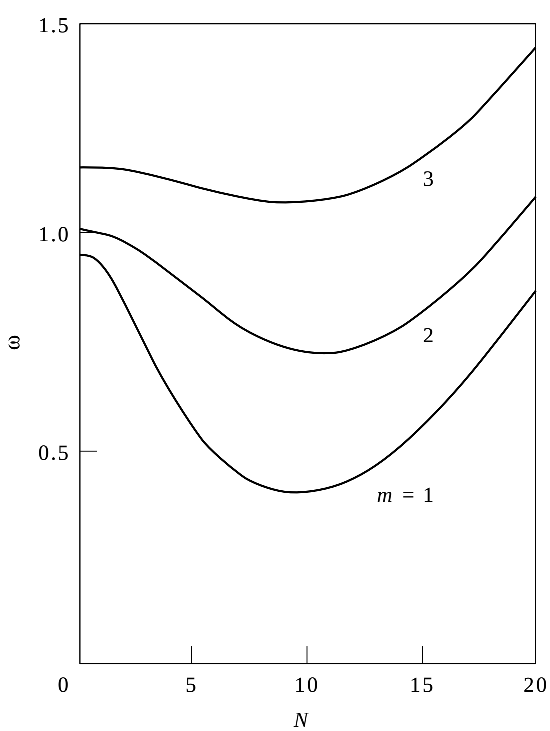

Figure 2. Natural frequency of liquidless cylinder: $Z = 50$.

where the b_m are unknown parameters. Substituting equations (61) and (62) into equation (37) and integrating, we obtain the general solution of $\bar{f}$ as:

$$\bar{f}(\xi) = C_1 \cosh \beta\xi + C_2 \sinh \beta\xi + C_3\, \beta\xi \cdot \cosh \beta\xi + C_4\, \beta\xi \cdot \sinh \beta\xi$$

$$-\sum_m \frac{b_m}{(\alpha_m^4 - \beta^4)^2} [2\beta^2\alpha_m^4\, \psi_m(\xi) + (\alpha_m^4 + \beta^4)\psi_m''(\xi)]$$

$$+\frac{\beta^2}{\alpha} \sum_n \sum_m \sum_e \frac{a_n\, b_m\, d_{nme}}{(\alpha_e^4 - \beta^4)^2} [(\alpha_e^4 + \beta^4)\psi_e(\xi) + 2\beta^2\psi_e''(\xi)], \tag{63}$$

$$d_{nme} = \int_0^\pi \psi_n''(\xi) \cdot \psi_m(\xi) \cdot \psi_e(\xi)\, \mathrm{d}\xi. \tag{64}$$

The unknown constants C_1 to C_4 can be determined with the boundary conditions (42) and (43).

So far, we have obtained expressions for w and f satisfying both the compatibility and the boundary conditions exactly.

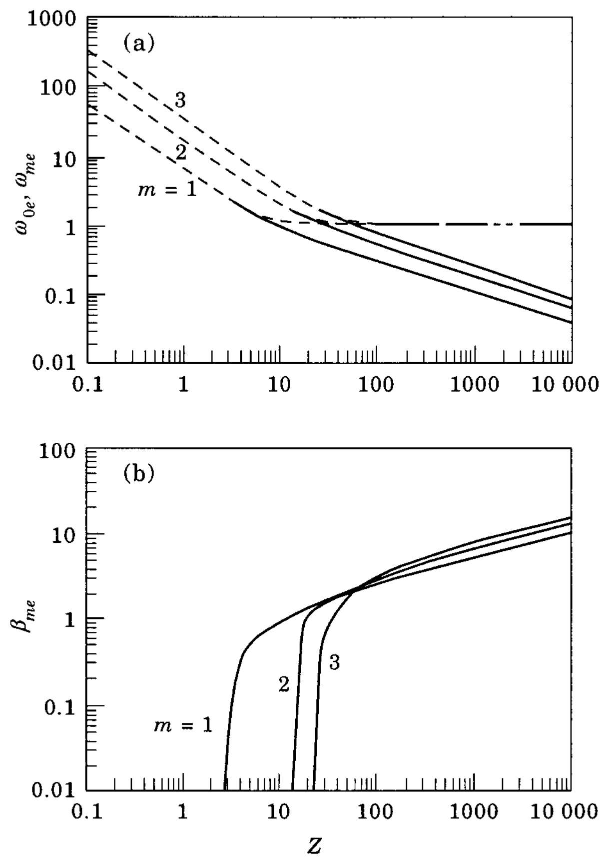

Figure 3. (a) Minimum natural frequency ω_{0e}, ω_{me}, and (b) corresponding wave number parameter β_{me}. ——, $N \neq 0$; -----, $N = 0$.

3.2.2. *Velocity potential*

We assume the velocity potential ϕ_j, satisfying the governing equation (43) and the boundary condition (44), as

$$\phi_j(\rho, \xi, \eta, \tau) = i\omega\, e^{i\omega\tau}\left[\sum_k A_{jk}\,\theta_{jNk} + \sum_l (B_{jl}\,\zeta_{jNl} + C_{jl}\,\chi_{jNl}) + D_j\,\rho^N + E\,\frac{1}{\rho^N}\right]\cos\eta \quad (65)$$

$$\theta_{jNk}(\rho, \xi) = G_{jNk}(\epsilon_{jNk}\,\rho)\cosh(\epsilon_{jNk}\,\tilde{\beta}\xi), \quad (66)$$

$$\zeta_{jNl}(\rho, \xi) = I_N\left(\frac{l\rho}{\tilde{\beta}l_j}\right)\cos\left(\frac{l}{l_j}\,\xi\right), \quad (67)$$

$$\chi_{jNl}(\rho, \xi) = K_N\left(\frac{l\rho}{\tilde{\beta}l_j}\right)\cos\left(\frac{l}{l_j}\,\xi\right) \quad (68)$$

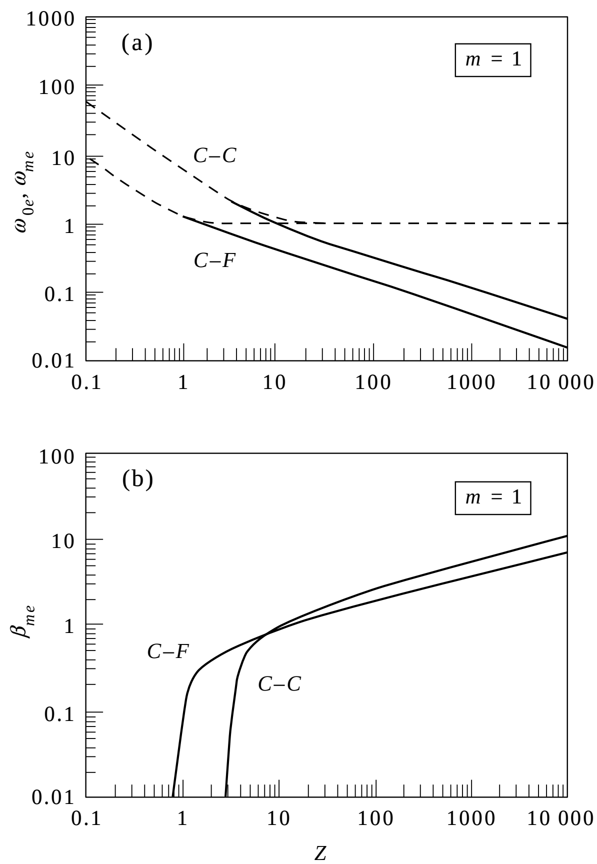

Figure 4. (a) Comparison of the minimum natural frequency, and (b) wave number parameter of clamped–clamped and clamped–free shell. ——, $N \neq 0$; -----, $N = 0$.

for the outer liquid ($j = o$),

$$G_{jNk}\,(\epsilon_{jNk}\,\rho) = Y_N\,(\epsilon_{jNk}\,\rho) - \frac{Y_{N},_{\rho}\,(\epsilon_{jNk})}{J_{N},_{\rho}\,(\epsilon_{jNk})}\,J_N\,(\epsilon_{jNk}\,\rho), \qquad j = \mathrm{O} \tag{69}$$

for the outer liquid with $j = o$, and

$$G_{jNk}\,(\epsilon_{jNk}\,\rho) = J_N\,(\epsilon_{jNk}\,\rho), \qquad j = i \tag{70}$$

for the inner liquid with $j = i$, where A_{jk}, B_{jl}, C_{jl}, D_j, and E are unknown parameters, and J_N, Y_N, I_N and K_N are the Bessel function of the first kind, Bessel function of the second kind, the modified Bessel function of the first, and the modified Bessel function of the second kind of order N, respectively. Furthermore, ϵ_{jNk} are the equation,

$$\left.\frac{\partial G_{jNk}\,(\epsilon_{jNk})}{\partial \rho}\right|_{\rho = 1\,\rho_o} = 0, \tag{71}$$

for the outer liquid ($j = o$), and

$$\left.\frac{\partial J_N\,(\epsilon_{jNk}\,\rho)}{\partial \rho}\right|_{\rho = 1} = 0, \tag{72}$$

for the inner liquid ($j = i$).

Hereafter, similar steps are employed as in reference 11. Using the boundary conditions

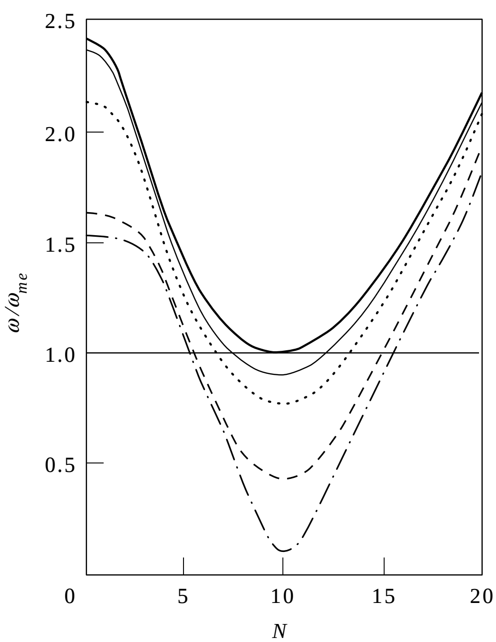

Figure 5. Natural frequency variation with compressive load: $Z = 50$, $m = 1$, without liquid. ——, $k_c = 0$; ——, $k_c = 10{\cdot}0$; ------, $k_c = 20{\cdot}0$; ——, $k_c = 30{\cdot}0$; –·–·–, $k_c = 31{\cdot}5$.

(46), and applying the Galerkin method, one can get the free-surface condition (45) in terms of A_{ik}, A_{ok}, and b_m in a matrix form as:

$$[(\mathbf{Q}_i \quad 0) - \omega^2(\mathbf{I} \quad \mathbf{S}_i)]\begin{Bmatrix} A_{ik} \\ b_m \end{Bmatrix} = \{0\},$$

$$[(\mathbf{Q}_o \quad 0) - \omega^2(\mathbf{I} \quad \mathbf{S}_o)]\begin{Bmatrix} A_{ok} \\ b_m \end{Bmatrix} = \{0\}. \tag{73}$$

Expressions $\mathbf{Q}_i$, $\mathbf{Q}_o$, $\mathbf{S}_i$ and $\mathbf{S}_o$ are presented in reference 11.

3.2.3. *Galerkin method*

Now, we shall seek values of b_m for the approximate satisfaction of the remaining governing equation (38). We apply the Galerkin method from which one can get a coupled equation in terms of A_{ik}, A_{ok}, and b_m as:

$$[(0 \quad 0 \quad \mathbf{E}) - \omega^2(\mathbf{G}_i \quad \mathbf{G}_o \quad \mathbf{F})]\begin{Bmatrix} A_{ik} \\ A_{ok} \\ b_m \end{Bmatrix} = \{0\}. \tag{74}$$

where $\mathbf{E}$ and $\mathbf{F}$ are the $1 \times m$ matrices while $\mathbf{G}_i$ and $\mathbf{G}_o$ are the 1×1 matrices. Actual expressions for the elements of these matrices are given in reference 11. Combining equations (73) and (74), we get

$$\left[\begin{pmatrix} \mathbf{Q}_i & 0 & 0 \\ 0 & \mathbf{Q}_o & 0 \\ 0 & 0 & \mathbf{E} \end{pmatrix} - \omega^2\begin{pmatrix} \mathbf{I} & 0 & \mathbf{S}_i \\ 0 & \mathbf{I} & \mathbf{S}_o \\ \mathbf{G}_i & \mathbf{G}_o & \mathbf{F} \end{pmatrix}\right]\begin{Bmatrix} A_{ik} \\ A_{ok} \\ b_m \end{Bmatrix} = \{0\}. \tag{75}$$

This is a coupled homogeneous linear equation in terms of A_{ik}, A_{ok} and b_m, from which one can obtain natural frequencies of the system as eigenvalues, and critical compressive load parameters which corresponds to zero natural frequency.

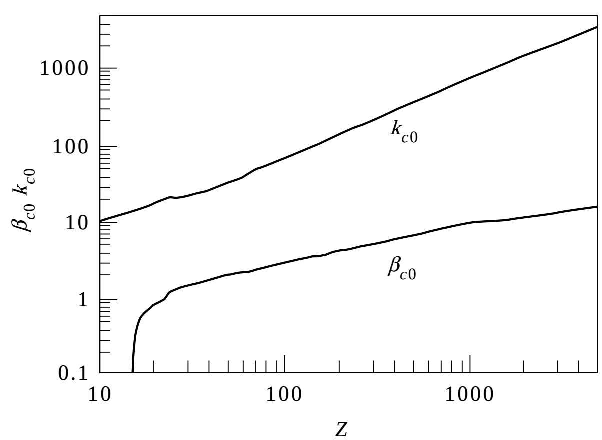

Figure 6. Axial compressive buckling load parameter k_{c0} and wave number β_{c0} of liquidless shell [12].

3.2.4. *Axisymmetric vibration*

In this case, the solution for w and ϕ_j are assumed as

$$w(\xi, \tau) = e^{i\omega\tau} \sum_m b_m \psi_m(\xi), \tag{76}$$

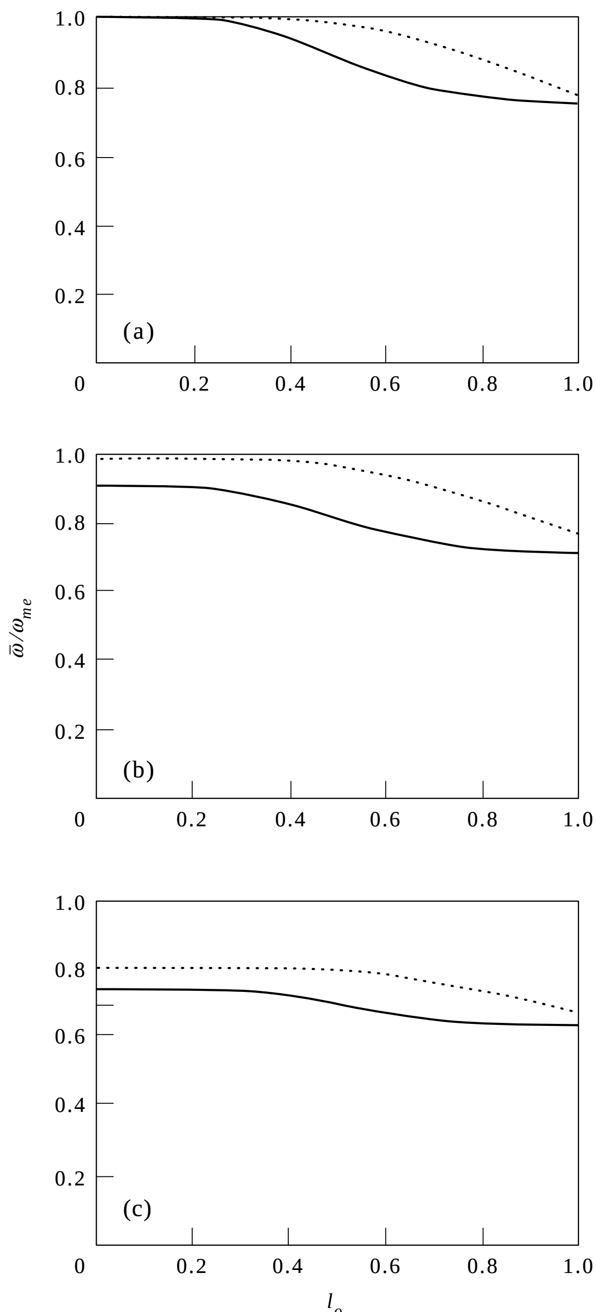

Figure 7. Minimum natural frequencies with outer liquid height l_o: $L/R = 0{\cdot}5$, $R/h = 50$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $m = 1$; (a) $l_i = 0$; (b) $l_i = 0{\cdot}5$; (c) $l_i = 1{\cdot}0$. ——, *C–C*; -----, *C–F*.

$$\phi_j\,(\rho,\,\xi,\,\tau) = i\omega\, e^{i\omega\tau}\left[\sum_k A_{jk}\,\theta_{jok} + \sum_l\,(B_{jl}\,\zeta_{jol} + C_{jl}\,\chi_{jol}) + D(\rho^2 + 2\tilde{\beta}^2\xi^2) + E\log\rho\right], \quad (77)$$

$$\theta_{jok}\,(\rho,\,\xi) = G_{jok}\,(\epsilon_{jok}\,\rho)\cosh\,(\epsilon_{jok}\,\tilde{\beta}\xi), \quad (78)$$

$$\chi_{jol}\,(\rho,\,\xi) = K_o\left(\frac{l\rho}{\tilde{\beta}l_j}\right)\cos\left(\frac{l}{l_j}\,\xi\right), \quad (79)$$

$$\zeta_{jol}\,(\rho,\,\xi) = I_o\left(\frac{l\rho}{\tilde{\beta}l_j}\right)\cos\left(\frac{l}{l_j}\,\xi\right), \quad (80)$$

$$G_{jok}\,(\epsilon_{jok}\,\rho) = -J_o\,(\epsilon_{jok}\,\rho) + \frac{J_1\,(\epsilon_{jok})}{Y_1\,(\epsilon_{jok})}\,Y_o\,(\epsilon_{jok}\,\rho), \quad (81)$$

$$\left.\frac{\partial G_{jok}\,(\epsilon_{jok})}{\partial\rho}\right|_{\rho\,=\,1\,\rho_o} = 0, \quad (82)$$

for the outer liquid with $j = o$, and

$$\phi_j\,(\rho,\,\xi,\,\tau) = i\omega\, e^{i\omega\tau}\left[\sum_k A_{jk}\,\theta_{jok} + \sum_l B_{jl}\,\zeta_{jlo} + B_o\,(\rho^2 - 2\tilde{\beta}^2\xi^2)\right], \quad (83)$$

$$\theta_{\,jok}\,(\rho,\,\xi) = J_o\,(\epsilon_{\,jok}\,\rho)\cosh\,(\epsilon_{\,jok}\,\tilde{\beta}\xi), \quad (84)$$

$$\zeta_{jol}\,(\rho,\,\zeta) = I_o\left(\frac{l\rho}{\tilde{\beta}l_j}\right)\cos\left(\frac{l}{l_j}\,\xi\right), \quad (85)$$

$$\left.\frac{\partial J_o\,(\epsilon_{\,jok}\,\rho)}{\partial\rho}\right|_{\rho\,=\,1} = 0 \quad (86)$$

for the inner liquid with $j = i$.

4. NUMERICAL RESULTS

Vibration characteristics and buckling strengths of the present liquid-shell coupled system are governed by the following system parameters: the thickness ratio R/h, the aspect ratio L/R, the density ratio $\bar{\rho}$, the radius ratio ρ_o, the material parameter $\bar{\bar{g}}$, and the liquid heights l_i, l_o. Among these parameters, we will be mainly concerned here with R/h, L/R and l_i, l_o, to clarify the influence of the liquids, inside and outside the shell, on the natural frequency and the buckling strength of the cylindrical shell system. For the engineering data from which one can presume the free vibration characteristics and buckling loads of a liquid-filled submerged shell, it would be convenient if the calculated results were normalized by those of the shell when the liquid heights are zero, i.e., without inner and outer liquids, as it had been done for the submerged cantilever cylinder [11].

In the calculations, unknown terms in equations (53), (61) and (65) were taken as $n = 20$, $m = 10$ and $k = l = 8$ to get reliable values as engineering data.

4.1. NATURAL VIBRATION OF LIQUIDLESS CLAMPED SHELL

Natural frequency variations of a liquidless clamped cylinder with circumferential wave number N are presented in Figure 2, as one example for $Z \equiv L^2\sqrt{1-\nu^2}/Rh = 50$ shell. From the figure, one finds that the minimum values for each axial vibration wave number

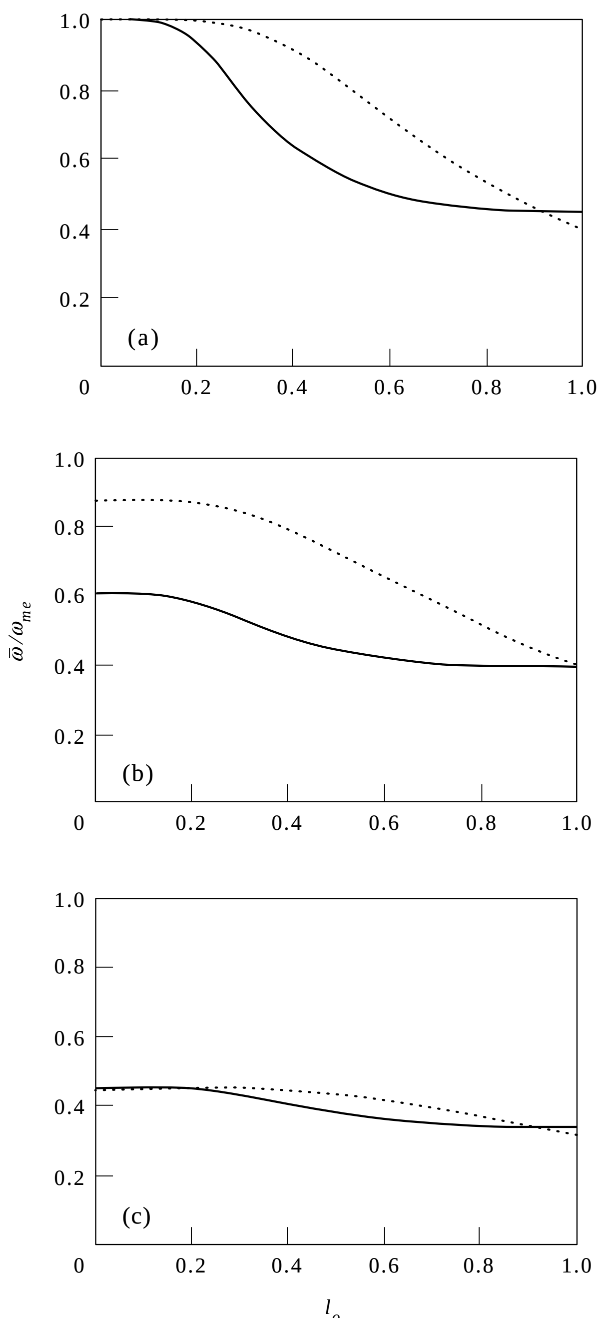

Figure 8. Minimum natural frequencies with outer liquid height l_o: $L/R = 0{\cdot}5$, $R/h = 500$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $m = 1$; (a) $l_i = 0$; (b) $l_i = 0{\cdot}5$; (c) $l_i = 1{\cdot}0$. ——, C–C; -----, C–F.

m correspond to that with $N \approx 10$. We hereafter consider those minimum natural frequencies with $m = 1 \sim 3$ modes.

For a wide range of the geometrical parameter Z, minimum natural frequency ω_{0e}, ω_{me} and corresponding wave number parameter β_{me} $(=LN/\pi R)$ has been obtained by Yamaki *et al*. [9] as shown in Figure 3. Present results without liquid agree with their results. From

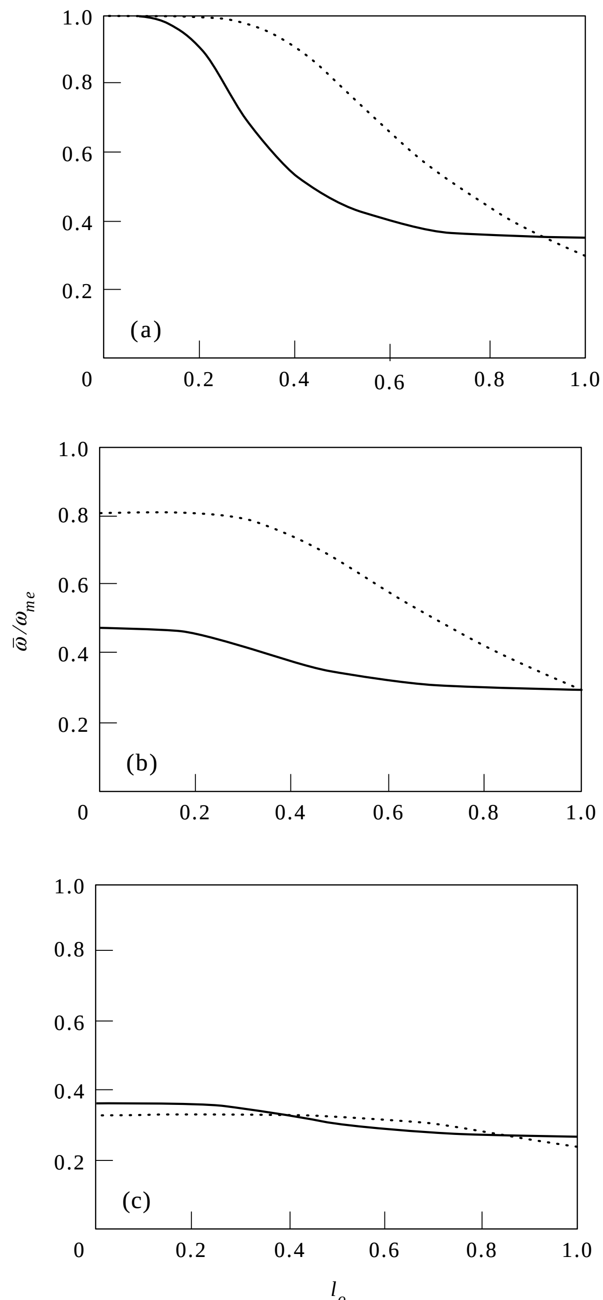

Figure 9. Minimum natural frequencies with outer liquid height l_o: $L/R = 2{\cdot}0$, $R/h = 500$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $m = 1$; (a) $l_i = 0$; (b) $l_i = 0{\cdot}5$; (c) $l_i = 1{\cdot}0$. ——, C–C; -----, C–F.

Figure 3, the minimum vibration mode is found to correspond to axisymmetric ($N = 0$) mode in a small range of Z.

To see the influence of the boundary condition of the upper end of the shell, comparisons with the results for a cantilever cylinder, i.e., clamped–free (C–F) are shown in Figure 4,

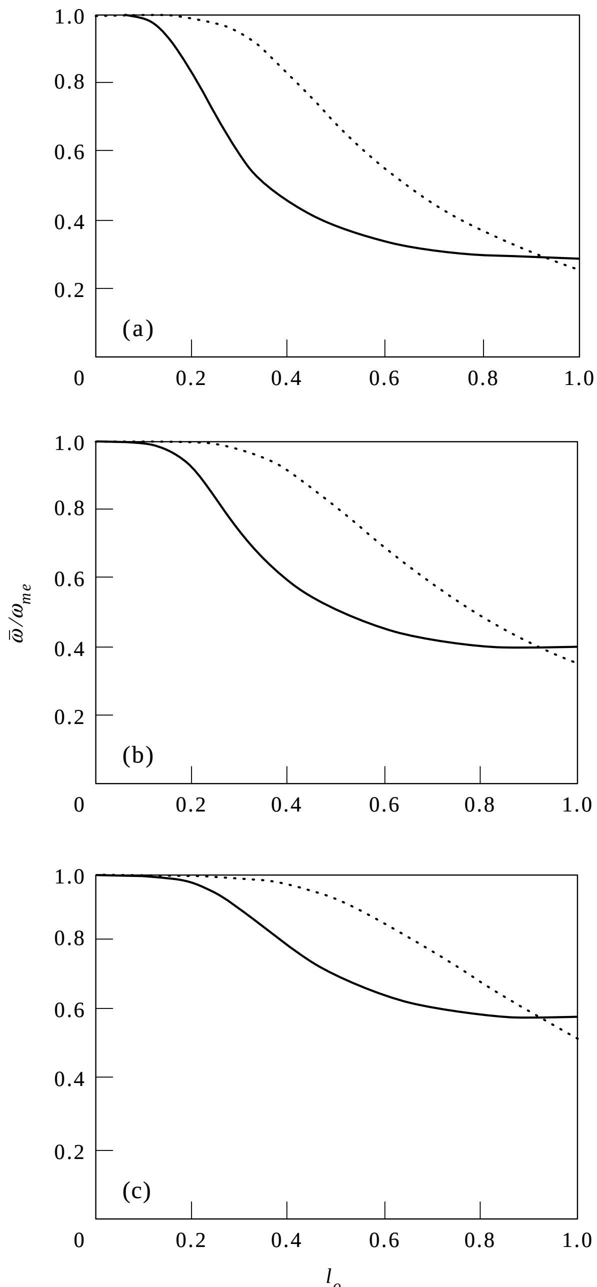

Figure 10. Effect of density ratio $\bar{\rho}$: $L/R = 2{\cdot}0$, $R/h = 100$, $\rho_o = 4{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_i = 0$, $m = 1$; (a) $\bar{\rho} = 1{\cdot}5$; (b) $\bar{\rho} = 3{\cdot}0$; (c) $\bar{\rho} = 8{\cdot}0$. ——, C–C; -----, C–F.

when $m = 1$. Restricting the motion of the upper end of the shell, the natural frequencies of a clamped–clamped shell are higher than those of a cantilever shell.

Hereafter, concerned with the natural frequency of the liquid-coupled shell, the normalized natural frequency of that of a liquidless shell, ω/ω_{me} will be used.

4.2. SHELL UNDER AXIAL COMPRESSIVE LOAD

Next, as an example of the influence of the axial compressive load on the natural frequency of the shell, natural frequency ratios ω/ω_{me} are shown in Figure 5 for the $Z = 50$ shell, when the axial compressive load parameter $k_c = 0, 10, 20, 30, 31{\cdot}5$. Natural frequency decreases with k_c, and it becomes zero which corresponds to the buckling. In this case, the critical value $k_{c0} = 31{\cdot}7$. For the liquidless shell, critical compressive load parameter k_{c0} and corresponding wave number parameter have been calculated by Yamaki and Kodama [12], as shown in Figure 6. The present results agree with theirs. Concerned with the compressive load parameter of the liquid-coupled shell, we will use the normalized compressive load parameter of that of the liquidless shell, $\overline{k}_c = k_c / k_{c0}$.

4.3. INFLUENCE OF SYSTEM PARAMETERS ON NATURAL FREQUENCY

Here, we consider the influence of the system parameters, i.e., R/h, L/R, $\bar{\rho}$, $\bar{\bar{g}}$, ρ_o, k_c, l_i and l_o, on the minimum natural frequency ratio $\bar{\omega}/\omega_{me}$. In the figures presented hereafter, as is mentioned above, the results for the C–F shell are also presented to see the difference of the boundary condition.

4.3.1. *Influence of thickness ratio R/h and aspect ratio L/R*

In Figure 7, frequency variations with the outer liquid height l_o are presented when the inner liquid height $l_i = 0$: (a), $l_i = 0{\cdot}5$: (b), $l_i = 1{\cdot}0$: (c), with $L/R = 0{\cdot}5$, $R/h = 50$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$ and $m = 1$. In this case, the shell is relatively thick and short.

From Figure 7(a), when the inner liquid height $l_i = 0$, the natural frequency decreases with the outer liquid height l_o. Reductions of the natural frequency are influenced more by l_o in the C–C case than in the C–F case. By filling liquid inside the shell, i.e., Figure 7(b) and (c), the values $\bar{\omega}/\omega_{me}$ at $l_o = 0$ are smaller than those of the $l_i = 0$ case, and the reductions of the natural frequency with l_o become small. When the shell becomes thinner, as shown in Figure 8, as $R/h = 500$, and taller as $L/R = 2{\cdot}0$ in Figure 9, the reduction of the natural frequency with l_o becomes large and the influence of the boundary condition

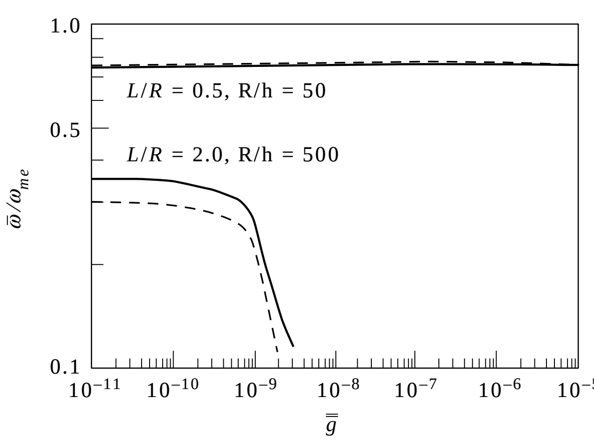

Figure 11. Minimum natural frequency with material parameter $\bar{\bar{g}}$: $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $l_i = 0$, $l_o = 1{\cdot}0$, $m = 1$. ——, C–C; -----, C–F.

is significant when the inner liquid is partially filled, i.e., $l_i = 0{\cdot}5$, and it becomes small for the full filled case where $l_i = 1{\cdot}0$.

4.3.2. *Influence of density ratio* $\bar{\rho}$

Influences of the density ratio $\bar{\rho}$ on the frequency variation with l_o are shown in Figure 10, where $\bar{\rho} = 1{\cdot}5$, $3{\cdot}0$ and $8{\cdot}0$ which correspond to the polyester/water, aluminum/water, steel/water cases. The reduction of the natural frequency becomes significant for smaller values of $\bar{\rho}$.

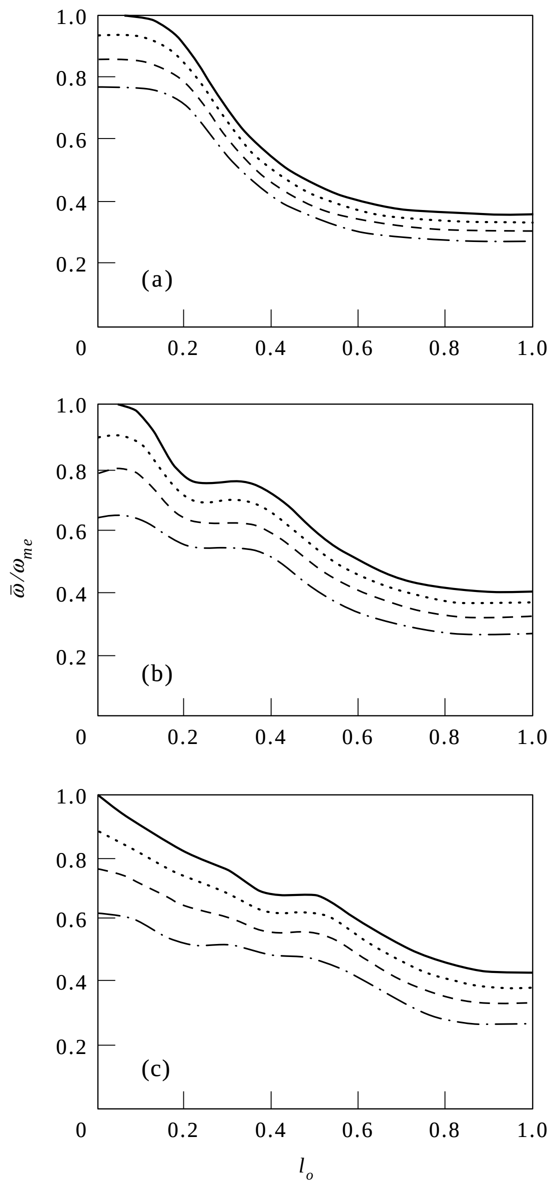

Figure 12. Influence of compressive load $\bar{k}_c$: $L/R = 2{\cdot}0$, $R/h = 500$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_i = 0$; (a) $m = 1$; (b) $m = 2$; (c) $m = 3$. ——, $\bar{k}_c = 0$; -----, $\bar{k}_c = 0{\cdot}25$; – – – –, $\bar{k}_c = 0{\cdot}50$; –·–·–·, $\bar{k}_c = 0{\cdot}75$.

4.3.3. *Influence of material parameter* $\bar{\bar{g}}$

Next, the influence of material parameter $\bar{\bar{g}}$ is considered. As can be expected from the definition of $\bar{\bar{g}} = gh\rho_s/E$, when $\bar{\bar{g}}$ becomes large, i.e., E becomes small and by keeping the others constant, the effect of static pressures both outside and inside the shell becomes relatively significant. In Figure 11, the variations of the fundamental natural frequency with $\bar{\bar{g}}$ are shown for $L/R = 0{\cdot}5$, $R/h = 50$ and $L/R = 2{\cdot}0$, $R/h = 500$.

The natural frequency is found to gradually decrease with an increase in $\bar{\bar{g}}$, and suddenly drops at a value of $\bar{\bar{g}}$ which corresponds to the buckling of the shell under an external liquid

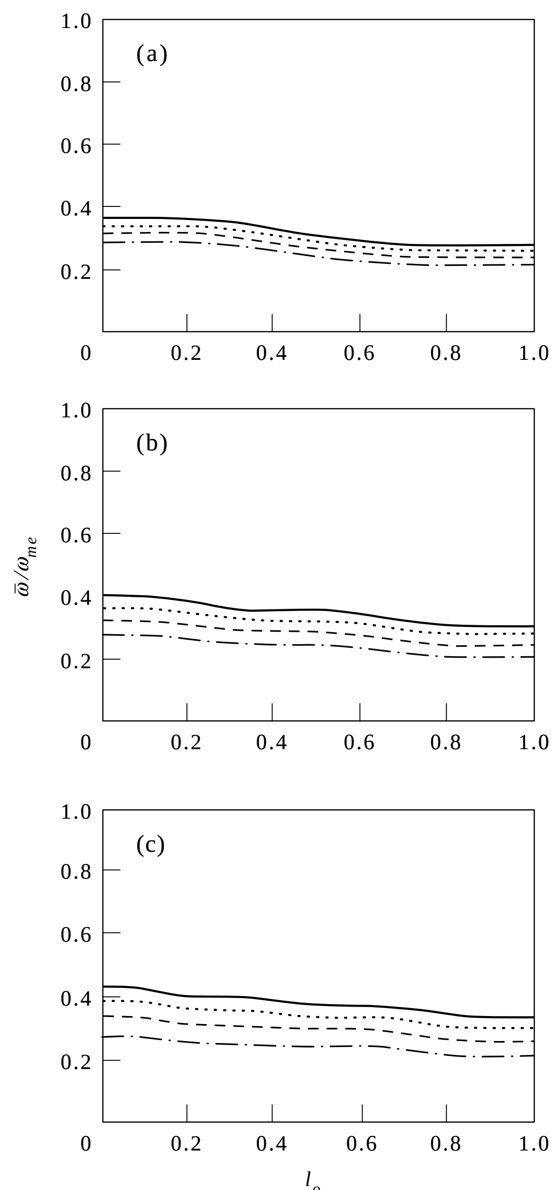

Figure 13. Influence of compressive load $\bar{k}_c$: $L/R = 2{\cdot}0$, $R/h = 500$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_i = 1{\cdot}0$; (a) $m = 1$; (b) $m = 2$; (c) $m = 3$. ——, $\bar{k}_c=0$; ----, $\bar{k}_c = 0{\cdot}25$; – – – –, $\bar{k}_c = 0{\cdot}50$; –·–·–, $\bar{k}_c = 0{\cdot}75$.

pressure. The degree of the frequency reduction is significant for a longer and thinner shell. Critical value of $\bar{\bar{g}}$ at which buckling occurs is larger in the C–C case than in the C–F case.

4.3.4. *Influence of axial compressive load parameter* $\bar{k}_c$

In Figure 5, the influence of axial compressive load parameter on the natural frequency variation when $l_i = 0$ has been presented. Next the case when the liquid is inside the shell is considered. The results when $\bar{k}_c = 0, 0{\cdot}25, 0{\cdot}5, 0{\cdot}75$ are shown in Figure 12 when $l_i = 0$, and in Figure 13 when $l_i = 1{\cdot}0$. Over the whole liquid range, l_o, the natural frequency is reduced by the axial compressive load.

4.4. VIBRATION MODE

Some examples for the vibration mode which correspond to the minimum natural frequency are shown in Figure 14, when $l_i = 0{\cdot}5$ and changing $l_o = 0, 0{\cdot}5, 1{\cdot}0$ for

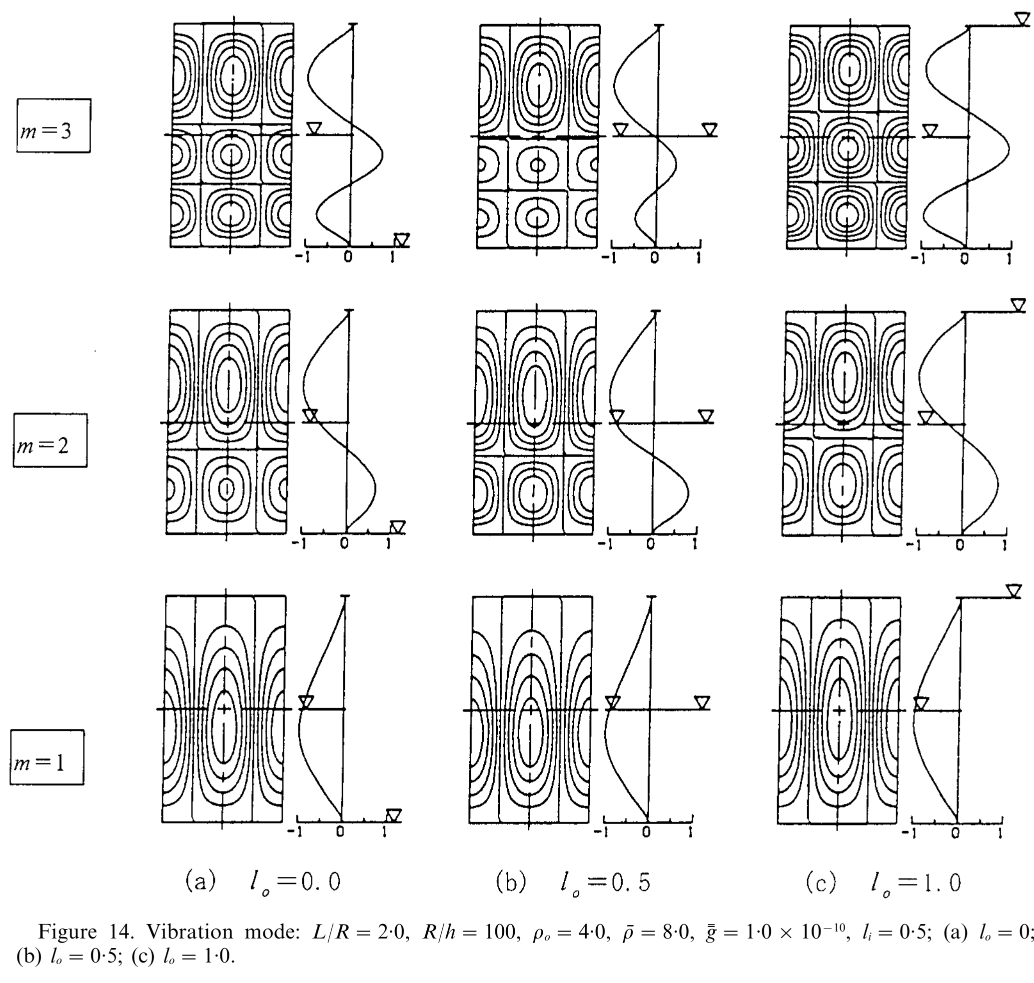

Figure 14. Vibration mode: $L/R = 2{\cdot}0$, $R/h = 100$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 8{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_i = 0{\cdot}5$; (a) $l_o = 0$; (b) $l_o = 0{\cdot}5$; (c) $l_o = 1{\cdot}0$.

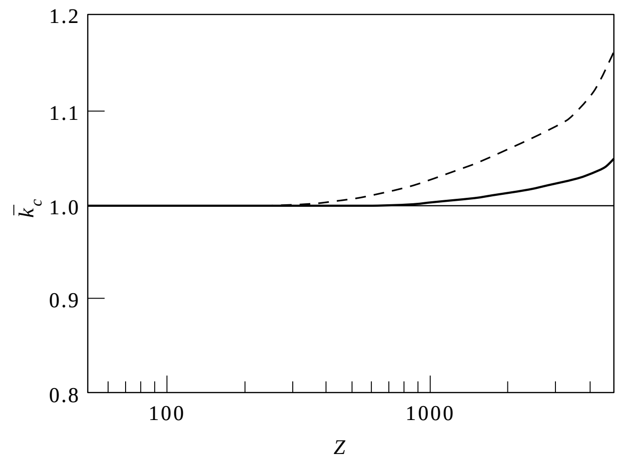

Figure 15. Influence of inner liquid height on buckling load $\bar{k}_c$: $\rho_o = 4{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_i = 1{\cdot}0$, $l_o = 0$. ——, $\bar{\rho} = 8{\cdot}0$; – – – –, $\bar{\rho} = 1{\cdot}5$.

$m = 1, 2, 3$. From the figure, the amplitude of the wall which does not face the liquid is found to be larger than that facing the liquid, for the mode with $m = 2, 3$.

4.5. BUCKLING UNDER COMPRESSIVE LOAD

Next, the influences of the inner and the outer liquids on the compressive buckling load ratio parameter $\bar{k}_c$ is studied.

First, the influence of the inner liquid is considered. Figure 15 represents $\bar{k}_c$ values with Z, when the inner liquid is fully filled and the outer liquid is absent. Inner liquid produces an outward hoop stress in the shell wall which strengthens the thin shell structure against the axial compressive load when compared with that of the liquidless shell. From the figure, the effect of the inner liquid is found to be significant for the shell with larger Z, i.e., for a longer and thinner shell, in which $\bar{k}_c$ is greater than unity.

Next, the influence of the outer liquid is considered. In this case, contrary to the inner liquid, the outer liquid produces an inward compressive hoop stress in the shell wall which

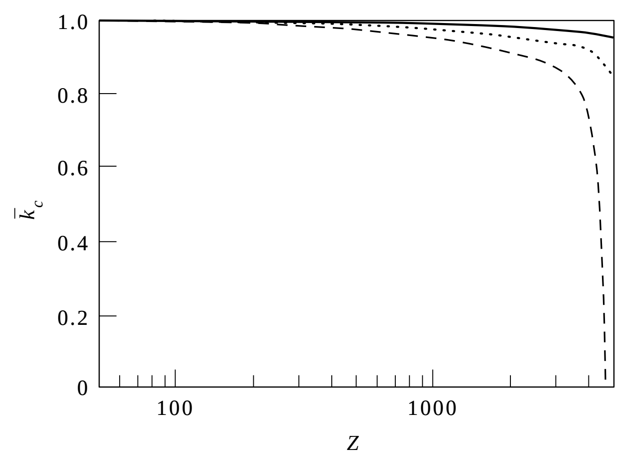

Figure 16. Influence of outer liquid height on buckling load $\bar{k}_c$: $\rho_o = 4{\cdot}0$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_i = 0$, $l_o = 1{\cdot}0$. ——, $\bar{\rho} = 8{\cdot}0$; ------, $\bar{\rho} = 3{\cdot}0$; – – – –, $\bar{\rho} = 1{\cdot}5$.

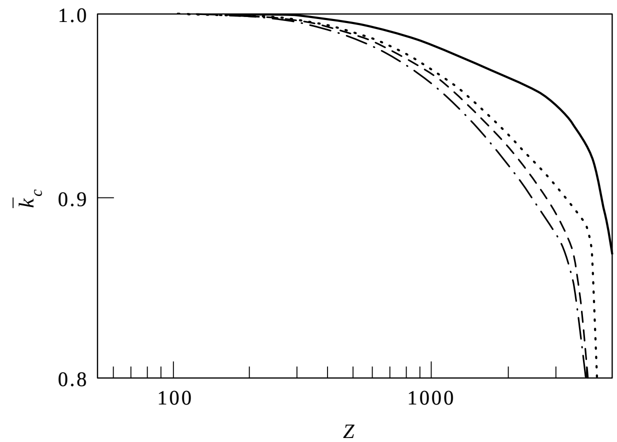

Figure 17. Influence of inner liquid height on buckling load $\bar{k}_c$: $\rho_o = 4{\cdot}0$, $\bar{\rho} = 1{\cdot}5$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, $l_o = 1{\cdot}0$, ——, $l_i = 0{\cdot}75$; -----, $l_i = 0{\cdot}50$; – – – –, $l_i = 0{\cdot}25$; –·–·–, $l_i = 0$.

weakens the shell structure against the axial compressive load. $\bar{k}_c$ values are presented in Figure 16 with Z when $l_o = 1{\cdot}0$, $l_i = 0$ and changing $\bar{\rho} = 8{\cdot}0, 3{\cdot}0, 1{\cdot}5$. By submerging, the compressive axial strength of the shell is found to be reduced. The influence of such outer liquid pressure is significant for the shell with large Z.

Finally, as some examples of the combined effect of the inner and the outer liquids, $\bar{k}_c$ are shown in Figure 17 when $\bar{\rho} = 1{\cdot}5$, $l_o = 1{\cdot}0$ and changing the inner liquid height as $l_i = 0, 0{\cdot}25, 0{\cdot}5, 0{\cdot}75$. By filling the inner liquid, buckling ratio $\bar{k}_c$ gradually increases in the range with larger values of Z. Figure 18 shows the results of the shell with $Z = 4500$. When the liquid is partially filled inside the shell ($l_i = 0{\cdot}5$), the degree of reduction in $\bar{k}_c$ with l_o becomes small, and when the liquid is fully filled in the shell ($l_i = 1{\cdot}0$), buckling strength becomes larger than the empty case as $\bar{k}_c > 1$, in the whole range of l_o.

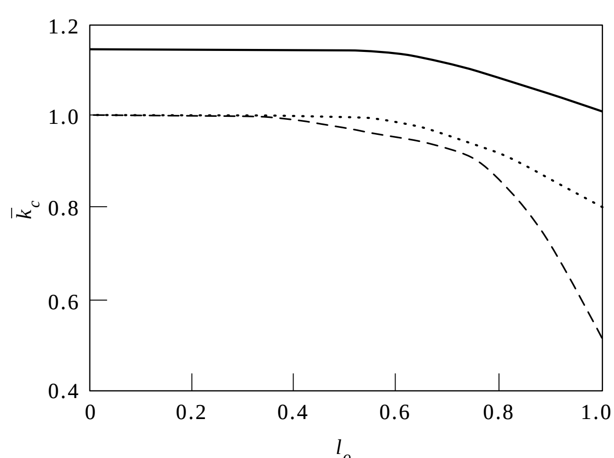

Figure 18. Influence of inner and outer liquids on buckling load $\bar{k}_c$: $Z = 4500$, $\rho_o = 4{\cdot}0$, $\bar{\rho} = 1{\cdot}5$, $\bar{\bar{g}} = 1{\cdot}0 \times 10^{-10}$, ——, $l_i = 1{\cdot}0$; -----, $l_i = 0{\cdot}5$; – – – –, $l_i = 0$.

5. COMPARISON WITH EXPERIMENTAL RESULTS

Finally, to confirm the validity of the theoretical analysis, experiments were conducted for the natural frequency of uncompressed shells by using a test cylinder made of polyester film with the geometrical parameter $Z = L^2\sqrt{1 - \nu^2}/Rh = 502$. Water was used as the liquid. The radius $R = 100$ mm, the thickness $h = 0{\cdot}244$ mm, the length $L = 113{\cdot}1$ mm, and the radius of the outer cylinder $R_o = 195$ mm. Detailed physical properties of the polyester film are shown in reference 11, Table I and details about the test equipment are also presented in reference 11.

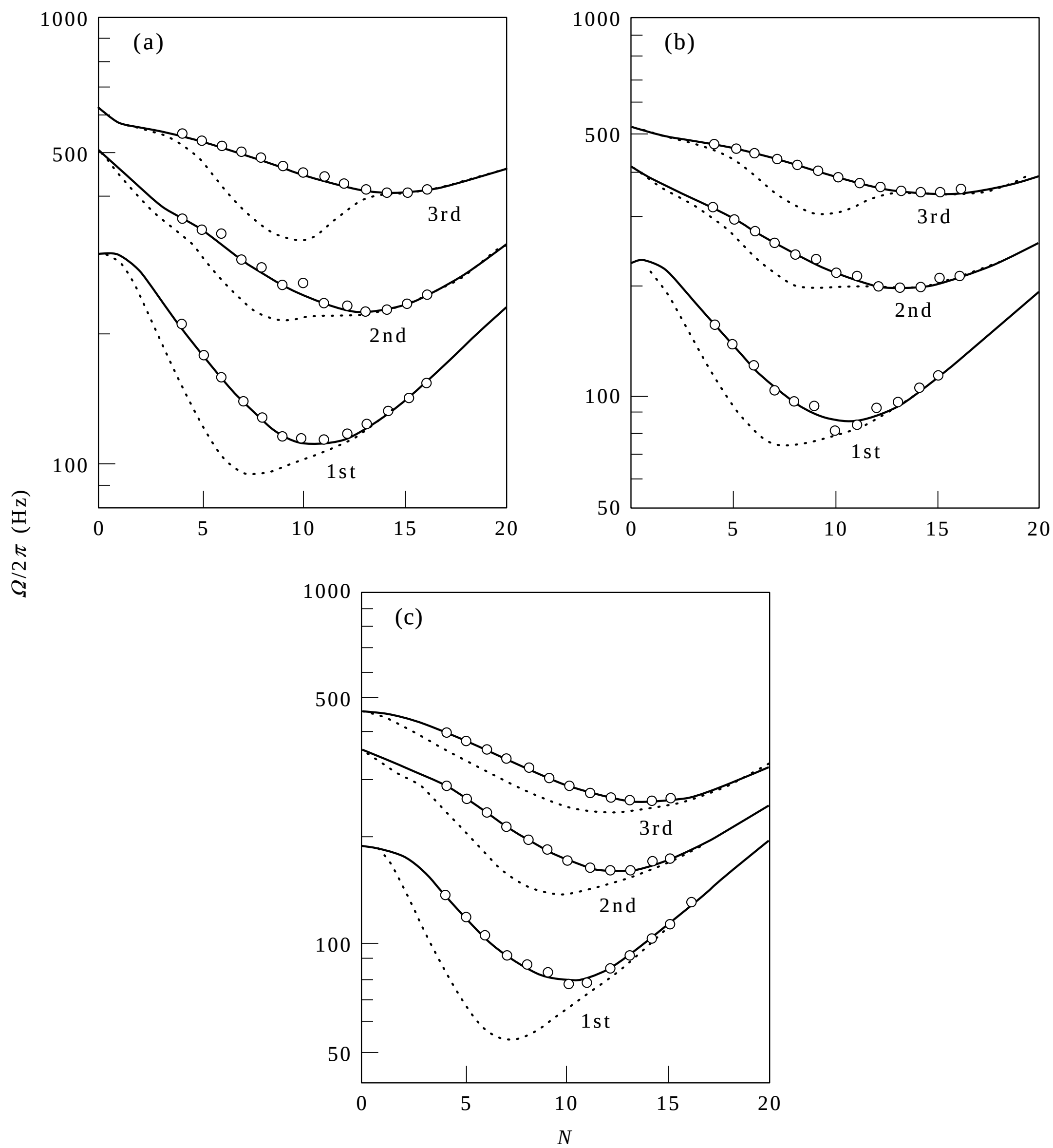

Figure 19. Comparison of natural frequency between numerical results and experiment, $Z = 502$, $l_o = 0{\cdot}5$; ——, theory (C–C); -----, theory (C–F); ○, experiment (C–C); (a) $l_i = 0{\cdot}25$; (b) $l_i = 0{\cdot}5$; (c) $l_i = 0{\cdot}75$.

In the experiment, the outer liquid height l_o was taken as 0·5, and the inner liquid height l_i was changed in 0·25 steps. The results are shown in Figure 19. In the figure, calculated results for the lower three axial vibration modes are shown with solid lines, while the experimental results are shown with circles. For reference, the theoretical results for the *C–F* shell are presented with dashed lines. One can see very good agreement between the theoretical results and experimental ones, which indicates the validity of the analysis.

6. CONCLUSION

A theoretical analysis has been carried out on the linear free vibration and the buckling under compression of a partially liquid containing thin clamped cylindrical shell that is also partially submerged in a liquid. In the analysis, the effect of the static liquid pressures on both the inside and outside surfaces of the shell was taken into account. The main results obtained from the present study are summarized as follows:

Natural frequency: (i) In general, the natural frequency of the shell decreases with the outer liquid height l_o or with the inner liquid height l_i. The degree of the reduction is significant for a thinner and longer shell. (ii) In the natural frequency variations with outer liquid height l_o, those of the *C–C* shell are influenced more by the outer liquid than those of the *C–F* shell, when the inner liquid is absent. (iii) For lower inner/outer liquid height, i.e., the partially filled/submerged case, the influence of outer/inner liquid height, l_o/l_i, on the natural frequency is large, while for higher inner/outer liquid height, the influence of outer/inner liquid height is small. In other words, when the shell is nearly fully filled/submerged, the reduction of the natural frequency with outer/inner liquid height, l_o/l_i, is very small. These come from the added mass effect. (iv) The experimental results are in good agreement with the theoretical results, which indicates the validity of the theoretical analysis.

Buckling strength: (v) By submerging, the axial strength of the shell decreases. The degree of strength reduction is significant for a shell with larger values of Z. (vi) The liquid inside the shell strengthens the shell under axial load. The effect is pronounced in a shell with large Z.

REFERENCES

1. H. Doki, N. Yamaki and J. Tani 1982 *Transactions of the Japan Society of Mechanical Engineers* (*Series A*) **48–434**, 1291–1299. Buckling of partially liquid-filled circular cylindrical shells under hydrostatic pressure.
2. H. Doki, N. Yamaki and J. Tani 1982 *Transactions of the Japan Society of Mechanical Engineers* (*Series A*) **48–434**, 1300–1309. Buckling of partially liquid-filled circular cylindrical shells under axial compression.
3. S. Kodama and N. Yamaki 1983 *Transactions of the Japan Society of Mechanical Engineers* (*Series A*) **49–439**, 366–376. Buckling of circular cylindrical shells under combined pressure and axial loadings.
4. M. Chiba, T. Yamashida and M. Yamauchi 1989 *Thin-Walled Structures* **8**(3), 217–233. Buckling of circular cylindrical shells partially subjected to external liquid pressure.
5. M. Chiba and S. Ubukata 1996 *Thin-Walled Structures* **24**(2), 113–122. Influence of internal liquid on buckling of circular cylindrical shells partially submerged in a liquid.
6. M. Chiba, N. Yamaki and J. Tani 1984 *Thin-Walled Structures* **2**(3), 256–284. Free vibration of a clamped–free circular cylindrical shell partially filled with liquid–part I: theoretical analysis.
7. M. Chiba, N. Yamaki and J. Tani 1984 *Thin-Walled Structures* **2**(4), 307–324. Free vibration of a clamped–free circular cylindrical shell partially filled with liquid–part II: numerical results.
8. M. Chiba, N. Yamaki and J. Tani 1985 *Thin-Walled Structures* **3**(1), 1–14. Free vibration of a clamped–free circular cylindrical shell partially filled with liquid–part III: experimental results.

9. N. Yamaki, J. Tani and T. Yamaji 1984 *Journal of Sound and Vibration* **94**(4), 531–550. Free vibration of a clamped–clamped circular cylindrical shell partially filled with liquid.
10. M. Chiba 1995 *Journal of the Acoustical Society of America* **97**(4), 2238–2248. Free vibration of a clamped–free circular cylindrical shell partially submerged in a liquid.
11. M. Chiba 1996 *Journal of the Acoustical Society of America* **100**(4), 2170–2180. Free vibration of a partially liquid-filled and partially submerged, clamped–free circular cylindrical shell.
12. N. Yamaki and S. Kodama 1972 *Report of Institute of High Speed Mechanics*, *Tohoku University* **25**–**245**, 61–98. Buckling of circular cylindrical shells under compression/report 3 (solutions based on the Donnell type equations considering prebuckling edge rotations).

APPENDIX: LIST OF SYMBOLS

c	parameter defined in equation (30)
d_{nme}	parameter defined by equation (64)
D	flexural rigidity of shell
E	Young's modulus
$F(f)$	stress function (non-dimensional form)
$g(\bar{g})$	gravitational acceleration
$\bar{\bar{g}}$	material parameter
$H_i\ (l_i = H_i/L)$	inner liquid height
$H_o\ (l_o = H_o/L)$	outer liquid height
h	shell thickness
k_c	axial compressive load parameter
$\bar{k}_c$	load ratio parameter
$L(=L/R)$	length of shell (aspect ratio)
m	axial mode number of vibration
$N(\beta)$	circumferential wave number
$P_{do}\ (p_{do})$	dynamic liquid pressure
$R(R/h)$	mean radius of shell
R_o	radius of outer shell
$t(\tau)$	time
$W(w)$	vibration amplitude of the shell
$W_o\ (w_o)$	static deflection of the shell
$x(\xi), y(\eta), z, r(\rho)$	co-ordinate system
v	Poisson's ratio
ρ_s, ρ_f	mass density of shell and liquid
$\bar{\rho}$	mass density ratio, $=\rho_s/\rho_f$
ρ_o	radius ratio, $=R_o/R$
$\Phi(\phi)$	velocity potential
$\psi_n\ (\xi)$	eigenfunction of beam defined by equation (54)
$\Omega(\omega)$	circular frequency
ω_{me}	asymmetric minimum natural frequency of liquidless shell
ω_{oe}	axisymmetric natural frequency of liquidless shell
$\bar{\omega}$	minimum natural frequency